4) a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Todas son diagonalizables ortogonalmente por ser matrices $\in \mathbb{R}^{n\times n}$ simétricas.

$P(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix}\right) = \lambda^2 - 2\lambda - 3$

Autovalores: $\boxed{\begin{matrix} \lambda_1 = 3 \\ \lambda_2 = -1 \end{matrix}}$

Para $\lambda = 3$

$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \xrightarrow{F2 \to F1+F2} \begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$ $x = y \to \bar{X} = x \cdot \boxed{\underbrace{(1,1)}_{\text{AVECT. } \lambda=3}}$ v_1

Para $\lambda = -1$.

$\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \xrightarrow{F2 \to F1-F2} \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix}$ $x = -y \to \bar{X} = y \cdot \boxed{\underbrace{(-1,1)}_{\text{AVECT. } \lambda=-1}}$ v_2

v_1 y v_2 son ortogonales. Los normalizo y armo U:

$\boxed{U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$ U ortogonal $\boxed{\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}$

tq: $\underline{A = U \Lambda U^T}$

b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $P(\lambda) = \det\left(\begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-1 \end{bmatrix}\right) =$

$= -(\lambda-1).(\lambda^2 - 2\lambda) = -\lambda^3 + 2\lambda^2 + \lambda^2 - 2\lambda = -\lambda^3 + 3\lambda^2 - 2\lambda.$

Autoval.: $\rightarrow \boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{matrix}}$

Para $\lambda = 0$

$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}_{F3 \to F1 - F3} \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{cases} -x - z = 0 \to x = -z \\ y = 0 \end{cases}$

$\to \bar{x} = z.\boxed{\underbrace{(-1, 0, 1)}_{\substack{\text{AVECT.} \\ \lambda = 0}}}$ $v1$

Para $\lambda = 1$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{cases} z = 0 \\ x = 0 \end{cases} \to \bar{x} = y.\boxed{\underbrace{(0, 1, 0)}_{\substack{\text{AVECT.} \\ \lambda = 1}}}$ $v2$

Para $\lambda = 2$

$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}_{F3 \to F1 + F3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{cases} x = z \\ y = 0 \end{cases} \to \bar{x} = z.\boxed{\underbrace{(1, 0, 1)}_{\substack{\text{AVECT.} \\ \lambda = 2}}}$ $v3$

Por prop. de matrices simétricas sabemos que $v1, v2$ y $v3$ son ortogonales.

Los normalizo y armo U ortogonal tq $A = U \Lambda U^T$

con $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\boxed{U = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}$